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MEASURE-THEORETIC COMPLEXITY OF ERGODIC SYSTEMS

BY

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ABSTRACT

We define an invariant of measure-theoretic isomorphism for dynamical systems, as the growth rate in n of the number of small $\bar{d}$-balls around α-n-names necessary to cover most of the system, for any generating partition α. We show that this rate is essentially bounded if and only if the system is a translation of a compact group, and compute it for several classes of systems of entropy zero, thus getting examples of growth rates in $O(n)$, $O(n^k)$ for $k \in \mathbb{N}$, or $o(f(n))$ for any given unbounded f, and of various relationships with the usual notion of language complexity of the underlying topological system.

In recent years, there has been a number of papers about the combinatorial notion of **symbolic complexity**, and its application to dynamical systems. Given a symbolic system, the symbolic complexity function $p(n)$ simply counts the number of words of length n in the language of the system; the function $p(n)$, or rather its rate of growth when n tends to infinity, is a topological invariant of the system, and moreover, when $p(n)$ has some simple forms (bounded, or sub-affine), then the system is fully known (see [FER2] for a longer discussion).

Is there a corresponding notion which is invariant by the (much weaker) notion of measure-theoretic isomorphism? Of course, it is always possible to code a given system (X, T, μ) into a symbolic system (X_α, T) by using a finite partition α, and to take the complexity of this system; but to get a measure-theoretic invariant, we

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should need the supremum of these functions over all measurable partitions. This, however, is generally not known, except if we restrict ourselves to some "good" partitions, with some regularity properties; hence, we need a notion which has a continuity property with respect to the usual metric on partitions; this leads to the idea of counting, not the number of different α-n-names, but the number of ϵ-$\bar{d}$-balls of α-n-names necessary to cover $(1-\epsilon)$ of the space. This gives a quantity $K(\alpha, n, \epsilon, T)$, which was defined first by Ratner ([RAT]), though with the distance $\bar{f}$ instead of $\bar{d}$; she computed its growth rate for different Cartesian powers of the horocycle flow, and showed that they are not Kakutani equivalent. More recently, while the present paper was in preparation, this invariant was also revived by Katok and Thouvenot ([KAT-THO]), who used it to build some actions of $\mathbb{Z}^2$ without Lipschitzian models.

Here, using the growth rates of the $K(\alpha, n, \epsilon, T)$, we define, up to equivalence when n tends to infinity, two functions $P_T^+(n)$ and $P_T^-(n)$, which are invariant by measure-theoretic isomorphism, and computable by using any generating partition, and may be seen as two measure-theoretic forms of the complexity function. The aim of this paper is not to use these invariants punctually to solve a given problem, but to present what should be (hopefully) the beginning of a general theory of measure-theoretic complexity.

This parallels the theory of symbolic complexity, as a particularly simple form of the invariants is equivalent to a simple characterization of the system: namely, our invariants are essentially bounded, in the sense that the $K(\alpha, n, \epsilon, T)$ are bounded in n for any ϵ, if and only if the system is isomorphic to a translation of a compact group. We show also that, as could be expected, the measure-theoretic complexities are (approximately) in e^{nh} whenever the system has entropy $h > 0$; so the interesting cases are to be found among systems of zero entropy. In this category, we have at our disposition some well-known groups of examples, the simplest (and, in the author's humble opinion, most useful) ones being the substitutions and the rank one systems; hence we proceed to give estimations for our invariants for these usual systems of entropy zero, and to compute them precisely for some sub-classes of systems; this helps to classify them up to measure-theoretic isomorphism, and gives examples of systems where the growth rates of the $K(\alpha, n, \epsilon, T)$ are in $O(n)$, $O(n^k)$ for $k \in \mathbb{N}$, or $o(f(n))$ for any given unbounded f; also, most of these systems having, under their symbolic forms, topological complexities in $O(n)$, we see that the measure-theoretic complexities may be equivalent to the topological complexity $p(n)$, or only in $O(p(n))$, or in $o((p(n))$ and unbounded, or in $o((p(n))$ and essentially bounded; we show also that our invariants are not complete for measure-theoretic isomorphism.

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1. Generalities

The setting of this paper is the one of **(finite) measure-preserving measurable dynamical systems** on Lebesgue spaces, $(X, \mathcal{A}, T, \mu)$. We refer the reader to [COR-FOM-SIN] for example, for any notion not explicitly defined in this paper. Equalities between measurable quantities are tacitly assumed to hold only **up to a set of measure zero. Each system will be assumed to be ergodic.**

Partitions are always assumed to be finite, and made with measurable sets; for technical reasons, we revive the Russian use to denote them by α and other Greek letters; the atoms of a partition α are denoted by $\alpha^1, \dots, \alpha^l$. For a partition α of X and a point $x \in X$, the **α-name** $\alpha(x)$ is the bi-infinite sequence $\alpha_n(x)$ where $\alpha_n(x) = i$ whenever $T^n x \in \alpha^i$. The **distance** between two ordered partitions $\alpha = \{\alpha^1, \dots, \alpha^l\}$ and $\beta = \{\beta^1, \dots, \beta^m\}$, is defined by

$$|\alpha - \beta| = \sum_{i=1}^{l \vee m} \mu(\alpha^i \Delta \beta^i),$$

the sets $\alpha^{l+1}, \dots, \beta^{m+1}, \dots$ being assumed to be empty.

A partition α **refines** a partition α' if each atom of α' is a union of atoms of α. We denote by $\alpha \vee \beta$ the upper bound of (α, β) for this partial order. A partition α is a **generating** partition for the system (X, T, μ) if the σ-algebra generated by $\bigvee_{n \in \mathbb{Z}} T^n \alpha$ separates all points except for a set of measure zero.

For two sequences $a = (a_1, \dots, a_k)$ and $b = (b_1, \dots, b_k)$ over a finite alphabet, we recall that

$$\bar{d}(a, b) = \frac{1}{k} \#\{i : a_i \neq b_i\}.$$

Let $\mathcal{H}$ be the set of all increasing functions from $\mathbb{N}$ to $\mathbb{N}$. Just to fix ideas, we choose, and denote by $\mathcal{H}_0$, some given scale of functions, for example $U(n) = an^b (\log n)^c e^{dn}$ for any a, b, c, d.

We can now proceed by steps towards the definition of measure-theoretic complexity:

Definition 1: For a point $x \in X$, we define

$$B(x, \alpha, n, \epsilon, T) = \{y \in X: \overline{d}((\alpha_0(x), \ldots, \alpha_{n-1}(x)), (\alpha_0(y), \ldots, \alpha_{n-1}(y))) < \epsilon\}.$$

And let $K(\alpha, n, \epsilon, T)$ be the smallest number K such that there exists a subset of X of measure at least $1 - \epsilon$ covered by at most K balls $B(x, \alpha, n, \epsilon, T)$.

What is dynamically significant is the growth rate of the $K(\alpha, n, \epsilon, T)$ for small ϵ, and we have to take some supremum of this on the set of partitions of X. There are several ways to do this: here, we try to get a precise enough invariant under a reasonably synthetic form, which inevitably leads us both to technicalities and abuses of notations.

Definition 2: For any $U \in \mathcal{H}$, we say that

$$P^+_{\alpha,T}(n) \prec U(n)$$

whenever

$$\lim_{\epsilon \to 0} \limsup_{n \to +\infty} \frac{K(\alpha, n, \epsilon, T)}{U(n)} \leq 1;$$

we say

$$P^+_{\alpha,T}(n) \succ U(n)$$

whenever

$$\lim_{\epsilon \to 0} \limsup_{n \to +\infty} \frac{K(\alpha, n, \epsilon, T)}{U(n)} \geq 1;$$

the union of these properties will be denoted as

$$P^+_{\alpha,T}(n) \sim U(n).$$

Definition 3: We say that

$$P^-_{\alpha,T}(n) \prec U(n)$$

whenever

$$\lim_{\epsilon \to 0} \liminf_{n \to +\infty} \frac{K(\alpha, n, \epsilon, T)}{U(n)} \leq 1;$$

we say

$$P^-_{\alpha,T}(n) \succ U(n)$$

whenever

$$\lim_{\epsilon \to 0} \liminf_{n \to +\infty} \frac{K(\alpha, n, \epsilon, T)}{U(n)} \geq 1;$$

the union of these properties will be denoted as

$$P^{-}_{\alpha,T}(n) \sim U(n).$$

Important remark: Our sign $\sim$ has to be interpreted carefully: if $P^{+}_{\alpha,T}(n) \sim U(n)$ and $P^{+}_{\alpha,T}(n) \sim V(n)$, the only thing we can say of U and V is that, if $U(n) \leq V(n)$ for all n large enough, then 1 must be an adherence value of $\frac{U(n)}{V(n)}$; in practice, this will be enough to distinguish whether $P^{+}_{\alpha,T}(n) \sim U(n)$ for any given U, hence, by abuse of language, any such function U will be said to be equal to $P^{+}_{\alpha,T}(n)$ up to equivalence when $n \to +\infty$; the same will be said for $P^{-}_{\alpha,T}(n)$, and the $P^{-}_{T}(n)$ and $P^{+}_{T}(n)$ defined below. Equivalently, we could suppress this ambiguity by taking our functions U only in $\mathcal{H}_0$; this is what we shall do in fact in this paper whenever we write a relation with $\sim$, but we preferred not to limit a priori the set of functions we may use.

Definition 4: The **upper measure-theoretic complexity** of the system, denoted by $P^{+}_{T}(n)$, is, up to equivalence when $n \to +\infty$ (see the important remark above), the

$$\sup_{\alpha} P^{+}_{\alpha,T}(n),$$

in the sense that $P^{+}_{T}(n) \prec U(n)$ if and only if $P^{+}_{\alpha,T}(n) \prec U(n)$ when α is any partition of X, and $P^{+}_{T}(n) \succ U(n)$ if and only if for any $V \in \mathcal{H}$ such that $V(n) \leq U(n)$ for n large enough and $\limsup_{n \to +\infty} \frac{V(n)}{U(n)} < 1$, there exists a partition α such that $P^{+}_{\alpha,T}(n) \succ V(n)$. So, again by abuse of notation, we write

$$P^{+}_{T}(n) \sim \sup_{\alpha} P^{+}_{\alpha,T}(n).$$

Definition 5: The **lower measure-theoretic complexity**, denoted by $P^{-}_{T}(n)$, is given by

$$P^{-}_{T}(n) \sim \sup_{\alpha} P^{-}_{\alpha,T}(n)$$

in the same sense.

PROPOSITION 1: *Every relation satisfied by $P^{+}_{T}(n)$ or $P^{-}_{T}(n)$, up to equivalence when $n \to +\infty$, is invariant by measure-theoretic isomorphism. In short, we say that the upper and lower measure-theoretic complexities are invariant by* *measure-theoretic isomorphism. This means concretely that whenever* $P_T^+(n) \prec U(n)$ *and* $P_S^+(n) \succ V(n)$, *with* $U(n) \leq V(n)$ *for every* n *large enough and* $\limsup_{n\to+\infty} \frac{U(n)}{V(n)} < 1$, *then* S *and* T *are not measure-theoretically isomorphic; and the same is true if we replace* P^+ *by* P^-.

Proof: Any measure-theoretic isomorphism transforms a measurable partition into another measurable partition. ∎

LEMMA 1: $P_T^+(n) \sim \sup_k P^+_{\alpha^{(k)},T}(n)$ *and* $P_T^-(n) \sim \sup_k P^-_{\alpha^{(k)},T}(n)$ *for any sequence of partitions* $\alpha^{(k)}$ *increasing to the whole* σ*-algebra* $\mathcal{A}$.

Proof: If $\alpha^{(k)}$ increase to $\mathcal{A}$, then for any measurable partition β and any $\epsilon > 0$, there exist a k and a subpartition γ of $\alpha^{(k)}$ such that $|\beta - \gamma| < \epsilon$. The Birkhoff ergodic theorem applied to the set $\bigcup_i \beta^i \Delta \gamma^i$ ensures that for almost every point x, and every n,

$$\bar{d}((\beta_0(x), \ldots, \beta_{n-1}(x)), (\gamma_0(x), \ldots, \gamma_{n-1}(x))) < \epsilon.$$

Hence, β being a fixed partition, for every ϵ, for almost every (depending on ϵ) x, for every k bigger than some $K(\epsilon)$, every δ and every n,

$$B(x, \beta, n, \delta + 2\epsilon, T) \supset B(x, \alpha^{(k)}, n, \delta, T),$$

which yields the result. ∎

COROLLARY 1: *If* α *is a generating partition for* (X, T), $P_T^+(n) \sim P^+_{\alpha,T}(n)$ *and* $P_T^-(n) \sim P^-_{\alpha,T}(n)$.

PROPOSITION 2:

$$\lim_{n\to+\infty} \frac{\log P_T^+(n)}{n} = \lim_{n\to+\infty} \frac{\log P_T^-(n)}{n} = h(T)$$

where $h(T)$ *is the measure-theoretic entropy of the system.*

Proof: By Shannon–Mc Millan–Breiman theorem (see for example [BIL]), whenever the partition α has entropy $h(\alpha, T) = h < +\infty$, then we can cover a subset of X of measure at least $1 - \epsilon$ by at most $e^{n(h+\epsilon)}$ sets E_i, such that $e^{-n(h+\epsilon)} < \mu(E_i) < e^{-n(h-\epsilon)}$, and for any $x \in E_i$, $y \in E_i$, $\alpha_j(x) = \alpha_j(y)$ for every $0 \leq j \leq n-1$; hence

$$K(\alpha, n, \epsilon, T) \leq e^{n(h+\epsilon)},$$

and also, for a given word w of length n, the number of words w' with $\bar{d}(w,w') < \epsilon$ is at most $\binom{n}{n\epsilon}(n\epsilon)^k \leq e^{ng(\epsilon)}$ for some $g(\epsilon) \to 0$ when $\epsilon \to 0$, and thence

$$K(\alpha, n, \epsilon, T) \geq (1-\epsilon)e^{n(h-\epsilon-g(\epsilon))}.$$

Hence $P_T^+(n)$ and $P_T^-(n)$ are dominated by every $e^{n(h+\epsilon)}$ and dominate every $e^{n(h-\epsilon)}$ if $h(T) = h > 0$, and $P_T^+(n)$ and $P_T^-(n)$ are dominated by every e^{dn}, $d > 0$, when $h(T) = 0$; for the same reason, $P_T^+(n)$ and $P_T^-(n)$ dominate every e^{dn}, $d > 0$, when $h(T) = +\infty$. ∎

Remark: When $h(T) = +\infty$, our invariants are not very interesting to compute; they satisfy the relations just above, but also, for any partition α with k elements, $P^+_{\alpha,T}(n) \prec k^n$, hence there is no hope, with finite partitions, to find growth rates in e^{n^2} for example. The domain of interest of the measure-theoretic complexities seem to be essentially restricted to systems of entropy zero.

LEMMA 2:

$$P^-_{S\times T}(n) \succ P^-_S(n)P^-_T(n)$$

and

$$P^+_{S\times T}(n) \prec P^+_S(n)P^+_T(n).$$

Proof: By Lemma 1 and because of the properties of the product measure, the measure-theoretic complexity of $S \times T$ may be computed by taking only partitions of the form $\alpha \times \beta = (\alpha^i \times \beta^j, 1 \leq i \leq k, 1 \leq j \leq l)$. But then

$$\begin{aligned}&B(x,\alpha,n,\epsilon,S) \times B(y,\beta,n,\epsilon,T)\\ \subset &B((x,y),\alpha\times\beta,n,2\epsilon,S\times T)\\ \subset &B(x,\alpha,n,2\epsilon,S)\times B(y,\beta,n,2\epsilon,T),\end{aligned}$$

which yields the result. ∎

2. Isometries

The simplest measure-preserving systems are the isometries of compact spaces, or equivalently the translations of compact (Abelian) groups, equipped with the Haar measure (see for example [FUR]); this includes both the irrational **rotations**, and the translations of the groups of p-adic integers, or p-adic **odometers**; this class of systems is completely characterized by its measure-theoretic complexity.

PROPOSITION 3: *T is measure-theoretically isomorphic to a translation of a compact group if and only if*

$$P_T^+(n) \prec U(n)$$

for every unbounded $U \in \mathcal{H}$, or if and only if

$$P_T^-(n) \prec U(n)$$

for every unbounded $U \in \mathcal{H}$.

Proof: Let T be an isometry of a compact space, equipped with the distance d, and α be a partition such that, if A_ϵ is the set

$$\{x\colon d(x, \delta\alpha^i) < \epsilon \text{ for some atom } \alpha^i\},$$

we have $\mu(A_\epsilon) < K\epsilon$ for some K.

We cover X by $L(\epsilon)$ open balls (for the distance d) of radius $\epsilon/2K$; then, if x and y are in the same ball, then $d(T^n x, T^n y) < \epsilon$ for every n; hence, applying the Birkhoff ergodic theorem to the set $A_{\epsilon/K}$, we get for each N big enough, after restricting x and y to a set of measure 1, that $T^n x$ and $T^n y$ lie in the same atom of α for at least $N(1-\epsilon)$ of the integers n between 0 and $N-1$; hence $K(\alpha, N, \epsilon, T) \leq L(\epsilon)$ for all N big enough, and hence the conclusion of the proposition is true for $P_{\alpha,T}^+(n)$, and hence also for $P_{\alpha,T}^-(n)$.

But such partitions α generate the whole σ-algebra, hence the "only if" part of our proposition.

Reciprocally, suppose $P_T^-(n)$ is dominated by any unbounded function, and let α be a partition. Then, for any $\epsilon < 1$, $K(\alpha, n, \epsilon, T)$ does not tend to infinity with n: indeed, if it does so for ϵ_0, we can choose M_p such that $K(\alpha, n, \epsilon_0, T) \geq p$ for all $n > M_p$, hence we have $K(\alpha, n, \epsilon, T) \geq p$ for all $n > M_p$ and $\epsilon \leq \epsilon_0$, and the function $U(n)$ taking the value p for $M_p \leq n < M_{p+1}$ contradicts the hypothesis.

Suppose now that T is not measure-theoretically isomorphic to an isometry. We may then choose a partition $\alpha = \{\alpha^0, \alpha^1\}$ independent of the Kronecker factor with $\mu(\alpha^i) > \frac{1}{3}$. Setting $\epsilon = \frac{1}{10}$, there exists a subset Γ of $\mathbb{N}$, with density one, such that $T^i\alpha$ is ϵ-independent of α for any i in Γ.

On the other hand, we choose some $K > \liminf_{n\to+\infty} K(\alpha, n, \epsilon^2/5, T)$; for some arbitrarily large N, $K(\alpha, N, \epsilon^2/5, T) \leq K$; we fix such an N, and cover $(1-\epsilon^2/5)$ of the space by balls $B_i = B(x_i, \alpha, N, \epsilon^2/5, T)$, $1 \leq i \leq L \leq K$. Then, for every $x \in B_i$, the set of n in $\{0, \ldots, N-1\}$ such that $\alpha_n(x) = \alpha_n(x_i)$

has cardinality at least $N(1-\epsilon^2/5)$; and hence, for every n inside a set $A_N \subset \{0,\ldots,N-1\}$ of cardinality at least $N(1-\epsilon)$, the set $\bigcup_{i=1}^{L}\{x \in B_i\colon \alpha_n(x) = \alpha_n(x_i)\}$ has measure at least $(1-\epsilon/5)$. But the L-uple $(\alpha_n(x_1),\ldots,\alpha_n(x_L))$ takes less than 2^K values while n ranges over A_N, so there exists a set $C_N \subset A_N$, with at most 2^K elements, such that for every $n \in A_N$, there exists $p(n) \in C_N$ such that $\alpha_n(x_i) = \alpha_{p(n)}(x_i)$ for every $1 \le i \le L$. Hence for $n \in A_N$,

$$|T^n\alpha - T^{p(n)}\alpha| = \int_{x\in X} |\alpha_n(x) - \alpha_{p(n)}(x)| < \epsilon.$$

It follows that there are $n_0 \in \{0,\ldots,N-1\}$ and $F_N \subset A_N$ such that the cardinality of F_N is greater than $N(1-\epsilon)/2^K$ and $|T^n\alpha - T^{n_0}\alpha| < \epsilon$ for $n \in F_N$. By translation we may then take $n_0 = 0$. For large N F_N must intersect Γ and, for $n \in F_N \cap \Gamma$, $T^n\alpha$ is both ϵ-independent and ϵ-close to α, which is a contradiction. ∎

Remark: This property, that $\bar{d}$-compacity is equivalent to isometry, is in fact a local property; the above proposition is still true if we ask to cover with balls $B(x,\alpha,n,\epsilon)$ only a subset of X of fixed measure $c > 0$.

3. Substitutions

For the last two parts, we recall some usual notions about symbolic dynamical systems.

For any sequence $u = (u_n, n \in \mathbb{N})$ on a finite alphabet A, we take T to be the one-sided shift and X the closure of the orbit of u under T; this defines the (topological) **symbolic system** associated to u.

A **word** is a finite string $w_1\cdots w_k$ of elements of A; the concatenation of two words w and w' is denoted by ww'. A word $w_1\cdots w_k$ is said to **occur** at place i in the sequence u if $u_i = w_1, \ldots, u_{i+k-1} = w_k$.

The **language** $L(u)$ is the set of all words occurring in u; the **complexity** of u is the function $p(n)$ which associates to each $n \in \mathbb{N}$ the number of words of length n in $L(u)$. By a slight abuse of notation (see the introduction, or [FER2]), we call it also the **symbolic complexity** of the associated topological system (X,T).

LEMMA 3: *If (X,T) is a symbolic system associated to a sequence u of complexity $p(n)$, and μ is a Borelian T-invariant measure, then (X,T,μ) satisfies*

$$P_T^-(n) \prec p(n)$$

and

$$P_T^+(n) \prec p(n).$$

Proof: For the generating partition in cylinders $x_0 = i$, every α-n-name must belong to the language of u, hence $K(\alpha,n,\epsilon,T) \leq p(n)$. ∎

We can then define the substitutions:

Definition 6: A **substitution** is an application from an alphabet A into the set $A^\star$ of finite words on A; it extends to a morphism of $A^\star$ for the concatenation by $\sigma(ww') = \sigma w \sigma w'$.

It is called **primitive** if there exists k such that a occurs in $\sigma^k b$ for any $a \in A$, $b \in A$.

It is called **of constant length** q if σa is of length q for any $a \in A$.

A **fixed point** of σ is an infinite sequence u with $\sigma u = u$.

The dynamical system associated to a primitive substitution is the symbolic system associated to any of its fixed points, equipped with its unique invariant probability; see [QUE] for more details.

For general primitive substitutions, we know that for any fixed point, $p(n)$ is smaller than some cn, $c > 0$ ([COB]), and hence the measure-theoretic complexities are at most in $O(n)$; we can give more precise results when the length is constant.

PROPOSITION 4: *Let σ be a primitive substitution of constant length q on a finite alphabet A, with non-periodic fixed points, (X,T,μ) the dynamical system associated to σ; let B be the alphabet we get by identifying a with b whenever $\bar{d}(\sigma^n a, \sigma^n b) \to 0$ when $n \to +\infty$; let τ be the substitution naturally defined by σ on B, and $p_\tau(n)$ the complexity of any fixed point of τ (this may then be computed by the algorithm given in [MOS]).*

Then τ is trivial (that is: B has only one letter) if and only if the dynamical system associated to σ is measure-theoretically isomorphic to a translation of a compact group. Whenever τ is not trivial,

$$P_T^+(n) \sim kn$$

where

$$0 < k = \limsup \frac{p_\tau(n)}{n},$$

and

$$P_T^-(n) \sim ln$$

where

$$0 < l = \liminf \frac{p_\tau(n)}{n}.$$

Proof: It is shown in [DEK], theorem 7, that whenever τ is trivial, then the system associated to σ is measure-theoretically isomorphic to a rotation, and in [LEM-MEN], lemma 8 (attributed to Host and Parreau), that the systems associated to the substitutions σ and τ are measure-theoretically isomorphic. Hence the complexities we have to compute are the same as those of the system (Y, T, ν) associated to τ.

Suppose now that τ is not trivial; then B has R letters, with $R \geq 2$, and there exists $c > 0$ such that $\overline{d}(\tau^n i, \tau^n j) > c$ whenever $i \in B$, $j \in B$, $i \neq j$; in particular, the fixed points of τ are not ultimately periodical. Then lemma 9 of [LEM-MEN] says that there exist $\delta > 0$ and $M \in \mathbb{N}$ such that, if u is a fixed point of τ, if w is the word $\tau^n(j_1)\cdots\tau^n(j_M)$ for some $n \in \mathbb{N}$, $j_1 \in B, \ldots, j_M \in B$, if w' is a word appearing in u at place p and if $\overline{d}(w, w') < \delta$, then $w = w'$ and p is a multiple of q^n. Note that in fact this lemma is not proved in [LEM-MEN]; it is an easy generalization of lemma 2.6 in [deJ], but only if we take into account the non-trivial result in [MOS], that a primitive substitution of constant length with a non ultimately periodical fixed point is recognizable.

Let $p(n)$, $n \geq 0$, be the complexity of the fixed point u. It is known ([QUE]) that $a_1 n < p(n) < a_2 n$ for n large enough, with $a_1 > 0$.

Let α be the generating partition of (Y, T) whose atoms are the cylinders $x_0 = i$, $i \in B$. We fix some ϵ small enough; then, for any n, $K(\alpha, n, \epsilon, T) \leq p_\tau(n)$.

We choose r such that $(M+2)q^r \leq n \leq (M+2)q^{r+1}$; for almost every point x of Y, the α-n-name $\alpha_0(x)\cdots\alpha_{n-1}(x)$ is a word of length n in the language of u, hence is of the form $f(x)w_1\cdots w_s d(x)$, where $M \leq s \leq q(M+2)$, the w_i are words of the form $\tau^r j_i$, $j_i \in B$, $f(x)$ is a final section of length $l_1(x)$ of some $\tau^r e(x)$, $e(x) \in B$, $d(x)$ is an initial section of length $l_2(x)$ of some $\tau^r h(x)$, $h(x) \in B$. Note that the α-n-name is determined by the parameters $n, j_1, \ldots, j_s$, $e(x)$, $h(x)$, and either $l_1(x)$ or $l_2(x)$.

Suppose now that $\epsilon < \delta/q(M+2)$ and that $y \in B(x,\alpha,n,\epsilon,T)$; lemma 9 of [LEM-MEN] implies that the α-n-name of y must be of the form $f(y)w_1\cdots w_s d(y)$, for any final section $f(y)$ of length $l_1(x)$ of any $\tau^r e(y)$, $e(y) \in B$, and any initial section $d(y)$ of length $l_2(x)$ of any $\tau^r h(y)$, $h(y) \in B$; then, either x and y have the same α-n-name, or $f(x)$ and $f(y)$ are two different words with $\overline{d}(f(x), f(y)) < \epsilon n/l_1(x)$, or $d(x)$ and $d(y)$ are two different words with $\overline{d}(d(x), d(y)) < \epsilon n/l_2(x)$.

Suppose that $f(x)$ and $f(y)$ are two different words with $\overline{d}(f(x), f(y)) < \epsilon n/l_1(x)$, and take k such that $c/q^{k+1} \leq q(M+2)\epsilon < c/q^k$. As $f(x)$ and $f(y)$ are final sections of words $\tau^r i$, if we cut them from the ends into words of length q^{r-k}, we get, for $z = x$ or y, $f(z) = f'(z)\tau^{r-k}j_1(z)\cdots\tau^{r-k}j_t(z)$, $f'(z)$ being a final section of $\tau^{r-k}j_0(z)$, or simply $f(z) = f'(z)$, in which case we put $t = 0$. And the hypotheses force $j_0(x) \neq j_0(y)$, $j_1(x) = j_1(y)$, ..., $j_t(x) = j_t(y)$. This implies that the words $\tau^k e(x)$ and $\tau^k e(y)$ have a common final section of exactly t letters, and this is true for at most one value (possibly zero) of t for each pair of values of $(e(x), e(y))$. As $l_1(x)$ must lie between tq^{r-k} and $(t+1)q^{r-k}$, this gives at most $R^2 q^{r-k}$ values for $l_1(x)$, and hence, for fixed n, no more than

$$R^2 q^{r-k} R^{qM+2q+2} \leq \frac{q^2}{c} R^{qM+2q+4} \epsilon n$$

possible different α-n names for x.

Taking into account the symmetric condition on $d(x)$ and $d(y)$, we have found some constant K of the system such that the number of possible α-n-names for a point x such that there exists y in $B(x,\alpha,n,\epsilon,T)$ with a different α-n-name is bounded by $K\epsilon n$ for every $\epsilon < \epsilon_0$ and every $n > N_0(\epsilon)$; also, for the unique T-invariant measure ν, there exist constants a_3 and a_4 such that, for any word w of length n in the language of u,

$$\frac{a_3}{n} < \nu(\{x\colon (\alpha_0(x),\ldots,\alpha_{n-1}(x)) = w\}) < \frac{a_4}{n}$$

(this is well-known, and easy to prove, first for $w = \tau^r a$, $a \in B$, for example by using the matrix of the substitution, and then for any w). Hence, if we want to cover a subset of Y of measure at least $1-\epsilon$ by balls $B(x,\alpha,n,\epsilon,T)$, we need to cover a subset of Y of measure at least $1-(Ka_4+1)\epsilon$ with balls reduced to one α-n-name, and we need at least $p_\tau(n) - (Ka_4+1)\frac{n}{a_3}\epsilon$ such balls; this minoration of $K(\alpha,n,\epsilon,T)$ yields the stated estimates; note that this implies in particular, because of our proposition 3, that, whenever τ is not trivial, the system is not isomorphic to a translation of a compact group. ∎

Example 1: The Morse system:

$$\begin{array}{lcl} a & \to & ab \\ b & \to & ba \end{array}$$

Then $\tau = \sigma$ (in fact, $c = 1$ and it is known since [deJ] that the Morse substitution satisfies lemma 9 of [LEM-MEN] with $L = 2$ and $\delta = \frac{1}{8}$) and we have only to compute $p(n)$. This can be computed by the algorithm given in [MOS], which yields $p(1) = 2, p(2) = 4, p(3) = 6$, and, for $n \geq 3, p(n+1) - p(n) = 2$ if $3.2^k < n \leq 4.2^k$, $p(n+1) - p(n) = 4$ otherwise. Hence finally we have

$$P_T^+(n) \sim \frac{10n}{3}$$

and

$$P_T^-(n) \sim 3n.$$

Example 2: The Rudin–Shapiro system:

$$\begin{array}{lcl} a & \to & ab \\ b & \to & ac \\ c & \to & db \\ d & \to & dc \end{array}$$

Then $\tau = \sigma$, $p_\tau(n) = 8n - 8$ for n large enough, and

$$P_T^+(n) \sim 8n \sim P_T^-(n).$$

Example 3:

$$\begin{array}{lcl} a & \to & abc \\ b & \to & bcd \\ c & \to & aba \\ d & \to & cda \end{array}$$

has a symbolic complexity satisfying $6n \leq p(n) \leq 7n$ for n large enough. Its measure-theoretic complexity is given by the symbolic complexity of the system associated to the "reduced" substitution

$$\begin{array}{lcl} a & \to & aba \\ b & \to & bac \\ c & \to & aca \end{array}$$

and satisfies $P_T^+(n) \sim kn$ for some $0 < k < 5$.

Remark: By taking suitable Cartesian products of substitutions with mutually prime constant lengths, we can build ergodic systems, which will have, by Lemma 2, measure-theoretic complexities in $O(n^k)$, for any natural integer k.

4. Rank one

Definition 7: A system (X, T, μ) is of rank one if for every partition α of X, for every positive ϵ, there exist a subset F of X, a positive integer h and a partition α' of X such that

- $F, TF, \dots, T^{h-1}F$ are disjoint,
- $|\alpha' - \alpha| < \epsilon$,
- α' is refined by the partition $(F, TF, \dots, T^{h-1}F, X - \bigcup_{i=0}^{h-1} T^i F)$.

We refer the reader to [FER3] for a general presentation of these fundamental examples of measure-preserving systems of entropy zero. For the general rank one system, we have just a majoration of one of the measure-theoretic complexities:

PROPOSITION 5: *For any rank one system* (X, T, μ),

$$P_T^-(n) \prec an^2$$

for any $a > 0$.

Proof: We take a partition α, and a sequence ϵ_n of real numbers decreasing to zero; by applying the definition of rank one to α and ϵ_n, we get sequences F_n, h_n, and α'_n; the non-atomicity of the space implies that $h_n \to +\infty$ and that $\eta_n = \mu(X - \bigcup_{i=0}^{h_n-1} T^i F_n) \to 0$ when $n \to +\infty$, and, by taking a subsequence, we may assume that h_n is increasing.

Let a be fixed, and, for a point x in F_n, let $s_n(x)$ be the smallest $i \geq 0$ such that $T^{h_n+i}x \in F_n$; we have $s_n(x) < ah_n$ on a subset G_n of F_n of relative measure at least $1 - \eta_n/a$; let $H_n = \bigcup_{i=0}^{h_n-1} T^i G_n$.

Let now y be a point of H_n; then, if $y = T^i x$, for $x \in G_n$, its α'_n-h_n-name depends only on i, which takes h_n values, and $s_n(x)$, which takes at most ah_n values; hence a fortiori

$$K(\alpha'_n, h_n, (1 + 1/a)\eta_n, T) < ah_n^2$$

and hence by a standard argument $K(\alpha, h_n, \sqrt{\epsilon_n} + (1+1/a)\eta_n, T) < ah_n^2$; which proves our proposition. ∎

To get a precise estimate, we need to know the exact definition of the system; we show how it works for the first and most famous example of rank one systems.

Definition 8: The Chacon system (X, T, μ) is the shift on the set of sequences (x_n), $n \in \mathbb{N}$, such that for every $s < t$ there exists m such that $x_s \cdots x_t$ is a subword of B_m, where $B_{n+1} = B_n B_n 1 B_n$, $B_0 = 0$, equipped with its unique invariant probability; see for example [FER1] for more details.

This definition implies that each x in X has a **canonical decomposition** into blocks B_n: for any m, $x_0 \cdots x_m$ is a concatenation of a suffix of B_n followed by blocks B_n separated by isolated 1, and any $m' > m$ gives the same decomposition on $x_0 \cdots x_m$; a block B_n in this decomposition is said to **occur in x in canonical position**.

LEMMA 4: *There exists $a > 0$ such that, if a word w occurs in some $x \in X$, with $\bar{d}(w, B_n) < a$, then $w = B_n$ and it occurs in the canonical position.*

Proof: We compute first $\bar{d}(0D_n, D_n0)$ where $B_n = 0D_n$; $\bar{d}(0D_1, D_10) = \frac{1}{2}$, then $D_{n+1} = D_n 0 D_n 1 0 D_n$, hence

$$h_{n+1}\bar{d}(0D_{n+1}, D_{n+1}0) = 2h_n\bar{d}(0D_n, D_n0) + h_n\bar{d}(0D_n, D_n1) + 1$$
$$= 3h_n\bar{d}(0D_n, D_n0) + 2$$

as D_n ends by a 0, hence $\bar{d}(0D_n, D_n0)$ is always greater than $\frac{1}{2}$.

Suppose now that the lemma is true for n with a replaced by $a_n < \frac{1}{8}$, and that its hypothesis is satisfied by w with a replaced by

$$a_{n+1} = a_n - \frac{1}{h_{n+1}}, \quad \text{where } h_n = \frac{3^{n+1}-1}{2} \text{ is the length of } B_n.$$

Then one of the three component words $w_0 \cdots w_{h_n-1}$, $w_{h_n} \cdots w_{2h_n-1}$ or $w_{2h_n+1} \cdots w_{3h_n}$ is a_n-close to B_n. Hence it is equal to B_n and occurs in canonical positions; but then the other component words can only be B_n or B_n shifted by one, but if any of them is shifted then $\bar{d}(w, B_{n+1}) > \frac{1}{7}$. Hence our result, starting from any $a_0 < 1$. ∎

PROPOSITION 6: *For the Chacon system,*

$$P_T^+(n) \sim 2n \sim P_T^-(n).$$

Proof: Let α by the generating partition into cylinders $x_0 = i$; the α-n-names are words of length n occurring in elements of X. It is shown in [FER1] that there are exactly $2n - 1$ such possible names.

Suppose that w and w' are two such n-names, with $\bar{d}(w, w') < \epsilon < a/10$ and suppose for example that $w = w_1 \cdots w_p B_n v_1 \cdots v_q$, $w_1 \cdots w_p$ being a suffix and $v_1 \cdots v_q$ a prefix of B_m; because of Lemma 4, w' can be only w, or $w_2 \cdots w_p 1 B_n v_1 \cdots v_q$, or $w_1 \cdots w_p B_n 1 v_1 \cdots v_{q-1}$, or $w_2 \cdots w_p 1 B_n 1 v_1 \cdots v_{q-1}$; but $\bar{d}(w_1 \cdots w_p, w_2 \cdots w_p 1) \geq \frac{1}{2}$ as we compare a concatenation of B_p with the same concatenation of B_p shifted by one, hence $w' = w_2 \cdots w_p 1 B_n v_1 \cdots v_q$ only if $p \leq 2\epsilon h_n$. A similar reasoning for the other cases and for other types of words w shows eventually that at most $10\epsilon n$ of the possible α-n-names may have some ϵ-$\bar{d}$-neighbours.

It also easy to see, for example by building the Rokhlin towers for T, that each α-n-name corresponds to an atom of measure lying between c_1/n and c_2/n; hence we conclude, like in Proposition 4, that

$$2n - Kn\epsilon \leq K(\alpha, n, \epsilon, T) \leq 2n,$$

and the proposition is proved. ∎

Corollary 2: *There exists a continuum of nonisomorphic systems (X, T, μ), disjoint in the sense of Furstenberg, with*

$$P_T^+(n) \sim 2n \sim P_T^-(n).$$

Proof: We take the systems $T_{(r_n)(s_n)}$ generated (in the same sense as Chacon's map) by the blocks B_n, where $B_0 = 0$ and $B_{n+1} = (B_n)^{r_n} 1 (B_n)^{s_n}$, with $r_n = 1$, $2 \leq s_n \leq L$ or $2 \leq r_n \leq L$, $s_n = 1$. By making straightforward modifications to the proofs of Lemma 4 and Proposition 6, we can prove that they have all the same complexities as Chacon's map, while $T_{(r_n)(s_n)}$ and $T_{(r'_n)(s'_n)}$ are disjoint except if $(r_n, s_n) = (r'_n, s'_n)$ for all n big enough ([FIE] or [deJ-RAH-SWA]). ∎

Another class of variants of Chacon's map gives interesting behaviours for the lower measure-theoretic complexity, particularly in view of the fact that the symbolic complexity of a system is either bounded or greater than n ([HED-MOR]):

PROPOSITION 7: *For any fixed unbounded function f, there exists a dynamical system which is weakly mixing (hence the lower measure-theoretic complexity must dominate some unbounded functions) with*

$$P_T^-(n) \prec f(n).$$

Proof: Let T be generated by the blocks B_n, where $B_0 = 0$ and $B_{n+1} = B_n^{s_n} 1 B_n^{s_n}$ for some sequence $s_n \to +\infty$. Let α be the partition into cylinders $x_0 = 0$ and $x_0 = 1$, and let h_n be the length of B_n.

For a given ϵ, let $L = L(\alpha, h_n, \epsilon, T) \geq K(\alpha, h_n, \epsilon, T)$ be such that the whole space X is covered by L balls $B(x_i, \alpha, h_n, \epsilon, T)$, $1 \leq i \leq L$. The α-h_n-name of x_i is some word, denoted by $w(a_i, e_i)$, made by a suffix of length $0 \leq a_i \leq h_n$ of B_n followed by $e_i = 0$ or $e_i = 1$ letter 1 and then followed by a prefix of B_n of length $h_n - a_i - e_i$.

Now, an $(\epsilon - 2/h_{n+1})$-dense set among all the possible α-h_{n+1}-names is made by all the $w(a_i, e_i)^k w(a_{j(i)}, e_{j(i)})^{s_n} w(a_{j'(i)}, e_{j'(i)})^{s_n - k}$, $1 \leq i \leq L$, $1 \leq k \leq s_n$, where $j(i)$ can take three values, i, $j_1(i)$ such that $w(a_i, e_i)$ shifted by 1 on the left is ϵ-close to $w(a_{j_1(i)}, e_{j_1(i)})$, and $j_2(i)$ the equivalent with left replaced by right; and $j'(i)$ takes values i, $j_1(i)$, $j_2(i)$, and $j_3(i)$, $j_4(i)$ defined in the same way with shifts by 2 instead of shifts by 1. But also

$$\bar{d}(w(a_i, e_i)^k w(a_j, e_j)^{s_n} w(a_{j'}, e_{j'})^{s_n - k}, w(a_i, e_i)^{k'} w(a_j, e_j)^{s_n} w(a_{j'}, e_{j'})^{s_n - k'}) < 2|k - k'|/s_n$$

for fixed i, j, j'. This allows one to have an arbitrarily slow growth of the $L(\alpha, h_n, \epsilon, T)$, and hence of the $K(\alpha, h_n, \epsilon, T)$, if s_n grows fast enough. ∎

Conjecture: We know that in general, for rank one systems defined as symbolic systems like in the proof of Proposition 5, the symbolic complexity $p(n)$ satisfies $\liminf_{n \to +\infty} p(n)/n^2 < +\infty$, but we may have $\limsup_{n \to +\infty} p(n)/n^k = +\infty$ for any k (see [FER2]). This behaviour should happen also for the measure-theoretic complexity, and we conjecture that the famous Ornstein mixing rank one transformation ([ORN]), while it must satisfy Proposition 5, has a $P_T^+(n)$ greater than $O(n^k)$ for any k.

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